



# Eigenvalues of the transversal Dirac operator on Kahler foliations

Georges Habib

## ► To cite this version:

Georges Habib. Eigenvalues of the transversal Dirac operator on Kahler foliations. Journal of Geometry and Physics, 2006, 56, pp.260-270. hal-00159109

**HAL Id: hal-00159109**

**<https://hal.science/hal-00159109>**

Submitted on 2 Jul 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Eigenvalues of the transversal Dirac Operator on Kähler Foliations

Georges Habib

Institut Élie Cartan, Université Henri Poincaré, Nancy I, B.P. 239

54506 Vandœuvre-Lès-Nancy Cedex, France

habib@iecn.u-nancy.fr

## Abstract

In this paper, we prove Kirchberg-type inequalities for any Kähler spin foliation. Their limiting-cases are then characterized as being transversal minimal Einstein foliations. The key point is to introduce the transversal Kählerian twistor operators.

## 1 Introduction

On a compact Riemannian spin manifold  $(M^n, g_M)$ , Th. Friedrich [Fri80] showed that any eigenvalue  $\lambda$  of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0, \quad (1.1)$$

where  $S_0$  denotes the infimum of the scalar curvature of  $M$ . The limiting case in (1.1) is characterized by the existence of a *Killing spinor*. As a consequence  $M$  is Einstein. K.D. Kirchberg [Kir86] established that, on such manifolds any eigenvalue  $\lambda$  satisfies the inequalities

$$\lambda^2 \geq \begin{cases} \frac{m+1}{4m} S_0 & \text{if } m \text{ is odd,} \\ \frac{m}{4(m-1)} S_0 & \text{if } m \text{ is even.} \end{cases}$$

On a compact Riemannian spin foliation  $(M, g_M, \mathcal{F})$  of codimension  $q$  with a bundle-like metric  $g_M$  such that the mean curvature  $\kappa$  is a basic coclosed

1-form, S.D. Jung [Jun01] showed that any eigenvalue  $\lambda$  of the transversal Dirac operator satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^\nabla, \quad (1.2)$$

where  $K_0^\nabla = \inf_M (\sigma^\nabla + |\kappa|^2)$ , here  $\sigma^\nabla$  denotes the transversal scalar curvature with the transversal Levi-Civita connection  $\nabla$ . The limiting case in (1.2) is characterized by the fact that  $\mathcal{F}$  is minimal ( $\kappa = 0$ ) and transversally Einstein (see Theorem 3.1). The main result of this paper is the following:

**Theorem 1.1** *Let  $(M, g_M, \mathcal{F})$  be a compact Riemannian manifold with a Kähler spin foliation  $\mathcal{F}$  of codimension  $q = 2m$  and a bundle-like metric  $g_M$ . Assume that  $\kappa$  is a basic coclosed 1-form, then any eigenvalue  $\lambda$  of the transversal Dirac operator satisfies:*

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla \quad \text{if } m \text{ is odd,} \quad (1.3)$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^\nabla \quad \text{if } m \text{ is even.} \quad (1.4)$$

The limiting case in (1.3) is characterised by the fact that the foliation is minimal and by existence of a transversal Kählerian Killing spinor (see Theorem 4.3). We refer to Theorem 4.4 for the equality case in (1.4).

We point out that Inequality (1.3) was proved by S. D. Jung [JK03] with the additional assumption that  $\kappa$  is *transversally holomorphic*. The author would like to thank Oussama Hijazi for his support.

## 2 Foliated manifolds

In this section, we summarize some standard facts about foliations. For more details, we refer to [Ton88], [Jun01].

Let  $(M, g_M)$  be a  $(p+q)$ -dimensional Riemannian manifold and a foliation  $\mathcal{F}$  of codimension  $q$  and let  $\nabla^M$  be the Levi-civita connection associated with  $g_M$ . We consider the exact sequence

$$0 \longrightarrow L \xrightarrow{\iota} TM \xrightarrow{\pi} Q \longrightarrow 0,$$

where  $L$  is the tangent bundle of  $TM$  and  $Q = TM/L \simeq L^\perp$  the normal bundle. We assume  $g_M$  to be a *bundle-like metric* on  $Q$ , that means the induced metric  $g_Q$  verifies the holonomy invariance condition,

$$\mathcal{L}_X g_Q = 0, \quad \forall X \in \Gamma(L),$$

where  $\mathcal{L}_X$  is the Lie derivative with respect to  $X$ . Let  $\nabla$  be the connection on  $Q$  defined by:

$$\nabla_X s = \begin{cases} \pi [X, Y_s], & \forall X \in \Gamma(L), \\ \pi (\nabla_X^M Y_s), & \forall X \in \Gamma(L^\perp), \end{cases}$$

where  $s \in \Gamma(Q)$  and  $Y_s$  is the unique vector of  $\Gamma(L^\perp)$  such that  $\pi(Y_s) = s$ . The connection  $\nabla$  is metric and torsion-free. The curvature of  $\nabla$  acts on  $\Gamma(Q)$  by :

$$R^\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]}s, \quad \forall X, Y \in \chi(M).$$

The transversal Ricci curvature is defined by:

$$\begin{aligned} \rho^\nabla : \Gamma(Q) &\longrightarrow \Gamma(Q) \\ X &\longmapsto \rho^\nabla(X) = \sum_{j=1}^q R^\nabla(X, e_j) e_j. \end{aligned}$$

Also, we define the transversal scalar curvature :

$$\sigma^\nabla = \sum_{i=1}^q g_Q(\rho^\nabla(e_i), e_i) = \sum_{i,j=1}^q R^\nabla(e_i, e_j, e_j, e_i),$$

where  $\{e_i\}_{i=1, \dots, q}$  is a local orthonormal frame of  $Q$  and  $R^\nabla(X, Y, Z, W) = g_Q(R^\nabla(X, Y)Z, W)$ , for all  $X, Y, Z, W \in \Gamma(Q)$ . The foliation  $\mathcal{F}$  is said to be transversally Einstein if and only if

$$\rho^\nabla = \frac{1}{q} \sigma^\nabla \text{Id},$$

with constant transversal scalar curvature. The mean curvature of  $Q$  is given by:

$$\kappa(X) = g_Q(\tau, X), \quad \forall X \in \Gamma(Q),$$

where  $\tau = \sum_{l=1}^p II(e_l, e_l)$ , with  $\{e_l\}_{l=1, \dots, p}$  is a local orthonormal frame of  $\Gamma(L)$  and  $II$  is the second fundamental form of  $\mathcal{F}$  defined by:

$$\begin{aligned} II : \Gamma(L) \times \Gamma(L) &\longrightarrow \Gamma(Q) \\ (X, Y) &\longmapsto II(X, Y) = \pi(\nabla_X^M Y). \end{aligned}$$

We define basic  $r$ -forms by :

$$\Omega_B^r(\mathcal{F}) = \{\Phi \in \Lambda^r T^*M \mid X \lrcorner \Phi = 0 \text{ and } X \lrcorner d\Phi = 0, \quad \forall X \in \Gamma(L)\},$$

where  $d$  is the exterior derivative and  $X_{\perp}$  is the interior product. Any  $\Phi \in \Omega_B^r(\mathcal{F})$  can be locally written as

$$\sum_{1 \leq j_1 < \dots < j_r \leq q} \beta_{j_1, \dots, j_r} dy_{j_1} \wedge \dots \wedge dy_{j_r},$$

where  $\frac{\partial}{\partial x_l} \beta_{j_1, \dots, j_r} = 0$ ,  $\forall l = 1, \dots, p$ . With the local expression of basic  $r$ -forms, one can verify that  $\kappa$  is closed if  $\mathcal{F}$  is isoparametric ( $\kappa \in \Omega_B^1(\mathcal{F})$ ). For all  $r \geq 0$ ,

$$d(\Omega_B^r(\mathcal{F})) \subset \Omega_B^{r+1}(\mathcal{F}).$$

We denote by  $d_B = d|_{\Omega_B(\mathcal{F})}$  where  $\Omega_B(\mathcal{F})$  is the tensor algebra of  $\Omega_B^r(\mathcal{F})$ . We have the following formulas:

$$d_B = \sum_{i=1}^q e_i^* \wedge \nabla_{e_i} \quad \text{and} \quad \delta_B = - \sum_{i=1}^q e_i \lrcorner \nabla_{e_i} + \kappa \lrcorner,$$

where  $\delta_B$  is the adjoint operator of  $d_B$  with respect to the induced scalar product and  $\{e_i\}_{i=1, \dots, q}$  is a local orthonormal frame of  $Q$ .

### 3 The transversal Dirac operator on Kähler Foliations

In this section, we start by recalling some facts on Riemannian foliations which could be found in [GK91a], [GK91b], [AG97], [Jun01]. For completeness, we also sketch a straightforward proof of Inequality ((1.2)) established in [Jun01] and end by recalling well-known facts (see [Kir86], [Kir96], [Hij94a], [Hij94b], [JK03]) on Kähler spin foliations.

On a foliated Riemannian manifold  $(M, g_M, \mathcal{F})$ , a transversal spin structure is a pair  $(\text{Spin}Q, \eta)$  where  $\text{Spin}Q$  is a  $\text{Spin}_q$ -principal fibre bundle over  $M$  and  $\eta$  a 2-fold cover such that the following diagram commutes:

$$\begin{array}{ccccc} \text{Spin}Q \times \text{Spin}_q & \longrightarrow & \text{Spin}Q & \longrightarrow & M \\ \downarrow \eta \otimes \text{Ad} & & \downarrow \eta & \nearrow & \\ \text{SO}Q \times \text{SO}_q & \longrightarrow & \text{SO}Q & & \end{array}$$

The maps  $\text{Spin}Q \times \text{Spin}_q \longrightarrow \text{Spin}Q$ , and  $\text{SO}Q \times \text{SO}_q \longrightarrow \text{SO}Q$ , are respectively the actions of  $\text{Spin}_q$  and  $\text{SO}_q$  on the principal fibre bundles  $\text{Spin}Q$  and

$SOQ$ . In this case,  $\mathcal{F}$  is called a transversal spin foliation. We define the foliated spinor bundle by:  $S(\mathcal{F}) := \text{Spin}Q \times_{\rho} \Sigma_q$ , where  $\rho : \text{Spin}_q \longrightarrow \text{Aut}(\Sigma_q)$ , is the complex spin representation and  $\Sigma_q$  is a  $\mathbb{C}$  vector space of dimension  $N$  with  $N = 2^{\lfloor \frac{q}{2} \rfloor}$ , where  $\lfloor \cdot \rfloor$  stands for the integer part. Recall that the Clifford multiplication  $\mathcal{M}$  on  $S(\mathcal{F})$  is given by:

$$\begin{aligned} \mathcal{M} : \Gamma(Q) \times \Gamma(S(\mathcal{F})) &\longrightarrow \Gamma(S(\mathcal{F})) \\ (X, \Psi) &\longmapsto X \cdot \Psi. \end{aligned}$$

There is a natural Hermitian product on  $S(\mathcal{F})$  such that, for all  $X, Y \in \Gamma(Q)$ , the following relations are true:

$$\begin{aligned} \langle X \cdot \Psi, \Phi \rangle &= -\langle \Psi, X \cdot \Phi \rangle, \\ X(\langle \Psi, \Phi \rangle) &= \langle \nabla_X \Psi, \Phi \rangle + \langle \Psi, \nabla_X \Phi \rangle, \\ \nabla_Y(X \cdot \Psi) &= (\nabla_Y X) \cdot \Psi + X \cdot (\nabla_Y \Psi), \end{aligned}$$

where  $\nabla$  is the Levi-Civita connection on  $S(\mathcal{F})$  and  $\Psi, \Phi \in \Gamma(S(\mathcal{F}))$ .

The transversal Dirac operator [GK91a, GK91b] is locally given by:

$$D_{tr} \Psi = \sum_{i=1}^q e_i \cdot \nabla_{e_i} \Psi - \frac{1}{2} \kappa \cdot \Psi, \quad (3.1)$$

for all  $\Psi \in \Gamma(S(\mathcal{F}))$ . We can easily prove using Green's theorem [YT90] that this operator is formally self adjoint. Furthermore, in [GK91b] it is proved that if  $\mathcal{F}$  is isoparametric and  $\delta_B \kappa = 0$ , then we have the Schrödinger-Lichnerowicz formula:

$$D_{tr}^2 \Psi = \nabla_{tr}^* \nabla_{tr} \Psi + \frac{1}{4} K_{\sigma}^{\nabla} \Psi,$$

where  $K_{\sigma}^{\nabla} = \sigma^{\nabla} + |\kappa|^2$  and

$$\nabla_{tr}^* \nabla_{tr} \Psi = - \sum_{i=1}^q \nabla_{e_i, e_i}^2 \Psi + \nabla_{\kappa} \Psi,$$

with  $\nabla_{X,Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$ , for all  $X, Y \in \Gamma(TM)$ . Denote by  $\mathcal{P}$  the transversal twistor operator defined by

$$\mathcal{P} : \Gamma(S(\mathcal{F})) \xrightarrow{\nabla_{tr}} \Gamma(Q^* \otimes S(\mathcal{F})) \xrightarrow{\pi} \Gamma(\ker \mathcal{M}),$$

where  $\pi$  is the orthogonal projection on the kernel of the Clifford multiplication  $\mathcal{M}$ . With respect to a local orthonormal frame  $\{e_1, \dots, e_q\}$ , for all  $\Psi \in \Gamma(S(\mathcal{F}))$ , one has

$$\mathcal{P}\Psi = \sum_{i=1}^q e_i^* \otimes (\nabla_{e_i} \Psi + \frac{1}{q} e_i \cdot D_{tr} \Psi + \frac{1}{2q} e_i \cdot \kappa \cdot \Psi). \quad (3.2)$$

For any spinor field  $\Psi$ , one can easily show that

$$\sum_{i=1}^q e_i \cdot \mathcal{P}_{e_i} \Psi = 0. \quad (3.3)$$

Now we give a simple proof of the following theorem:

**Theorem 3.1** [Jun01] *Let  $(M, g_M, \mathcal{F})$  be a compact Riemannian manifold with a spin foliation  $\mathcal{F}$  of codimension  $q$  and a bundle-like metric  $g_M$  with  $\kappa \in \Omega_B^1(\mathcal{F})$ . Assume that  $\delta_B \kappa = 0$  and let  $\lambda$  be an eigenvalue of the transversal Dirac operator, then*

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^\nabla. \quad (3.4)$$

**Proof.** For all  $\Psi \in \Gamma(S(\mathcal{F}))$ , we have using Identities (3.2), (3.3) (3.1),

$$|\mathcal{P}\Psi|^2 = |\nabla^{tr} \Psi|^2 - \frac{1}{q} |D_{tr} \Psi|^2 - \frac{1}{q} \Re(D_{tr} \Psi, \kappa \cdot \Psi) - \frac{1}{4q} |\kappa|^2 |\Psi|^2.$$

For any spinor field  $\Phi$ , we have that  $(\Phi, \kappa \cdot \Phi) = -(\kappa \cdot \Phi, \Phi) = -\overline{(\Phi, \kappa \cdot \Phi)}$ , so the scalar product  $(\Phi, \kappa \cdot \Phi)$  is a pure imaginary function. Hence for any eigenspinor  $\Psi$  of the transversal Dirac operator, we obtain

$$\int_M |\mathcal{P}\Psi|^2 + \frac{1}{4q} \int_M |\kappa|^2 |\Psi|^2 = \int_M |\nabla^{tr} \Psi|^2 - \frac{1}{q} \int_M \lambda^2 |\Psi|^2,$$

from which we deduce (3.4) with the help of the Schrödinger-Lichnerowicz formula. Finally, we can easily prove in the limiting case that  $\mathcal{F}$  is minimal i.e.  $\kappa = 0$ , and transversally Einstein.  $\square$

A foliation  $\mathcal{F}$  is called Kähler if there exists a complex parallel orthogonal structure  $J : \Gamma(Q) \rightarrow \Gamma(Q)$  ( $\dim Q = q = 2m$ ). Let  $\Omega$  be the associated Kähler, i.e., for all  $X, Y \in \Gamma(Q)$ ,  $\Omega(X, Y) = g_Q(J(X), Y) = -g_Q(X, J(Y))$ . The Kähler form can be locally expressed as

$$\Omega = \frac{1}{2} \sum_{i=1}^q e_i \cdot J(e_i) = -\frac{1}{2} \sum_{i=1}^q J(e_i) \cdot e_i,$$

and for all  $X \in \Gamma(Q)$ , we have  $[\Omega, X] := \Omega \cdot X - X \cdot \Omega = 2J(X)$ . Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum

$$S(\mathcal{F}) = \bigoplus_{r=0}^m S_r(\mathcal{F}),$$

where  $S_r(\mathcal{F})$  is an eigenbundle associated with the eigenvalue  $i\mu_r = i(2r - m)$  of the Kähler form  $\Omega$ . Moreover, the spinor bundle of a Kähler spin foliation carries a parallel anti-linear map  $j$  satisfying the relations:

$$\begin{aligned} j^2 &= (-1)^{\frac{m(m+1)}{2}} Id, \\ [X, j] &= 0, \\ (j\Psi, j\Phi) &= (\Phi, \Psi), \end{aligned}$$

and we have  $j\Psi_r = (j\Psi)_{m-r}$ . For all  $X \in \Gamma(Q)$ , we have

$$p_+(X) \cdot S_r(\mathcal{F}) \subset S_{r+1}(\mathcal{F}) \quad \text{and} \quad p_-(X) \cdot S_r(\mathcal{F}) \subset S_{r-1}(\mathcal{F}),$$

where  $p_{\pm}(X) = \frac{X \mp iJ(X)}{2}$ . We define the operator  $\tilde{D}_{tr}$  by

$$\tilde{D}_{tr}\Psi = \sum_{i=1}^q J(e_i) \cdot \nabla_{e_i} \Psi - \frac{1}{2} J(\kappa) \cdot \Psi.$$

The local expression of  $\tilde{D}_{tr}$  is independant of the choice of the local frame and by Green's theorem [YT90], we prove that this operator is self-adjoint. On a Kähler spin foliation, the operators  $D_{tr}$  and  $\tilde{D}_{tr}$  satisfy:

$$[\Omega, D_{tr}] = 2\tilde{D}_{tr}, \tag{3.5}$$

$$[\Omega, \tilde{D}_{tr}] = -2D_{tr}, \tag{3.6}$$

$$[\Omega, D_{tr}^2] = 0, \tag{3.7}$$

$$D_{tr}\tilde{D}_{tr} + \tilde{D}_{tr}D_{tr} = 0, \tag{3.8}$$

$$\tilde{D}_{tr}^2 = D_{tr}^2. \tag{3.9}$$

We should point out that Equations (3.7), (3.8) and (3.9) are true under the assumptions that  $\mathcal{F}$  is isoparametric and  $\delta_B \kappa = 0$ . Now we define the two operators  $D_+$  and  $D_-$  by

$$D_+ = \frac{1}{2}(D_{tr} - i\tilde{D}_{tr}) \quad \text{and} \quad D_- = \frac{1}{2}(D_{tr} + i\tilde{D}_{tr}). \tag{3.10}$$

Furthermore,  $D_{tr}$  splits into  $D_+$  and  $D_-$ , and we have the two exact sequences:

$$\Gamma(S_m(\mathcal{F})) \xrightarrow{D_-} \dots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_-} \Gamma(S_{r-1}(\mathcal{F})) \xrightarrow{D_-} \dots \Gamma(S_0(\mathcal{F})), \tag{3.11}$$

$$\Gamma(S_0(\mathcal{F})) \xrightarrow{D_+} \dots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_+} \Gamma(S_{r+1}(\mathcal{F})) \xrightarrow{D_+} \dots \Gamma(S_m(\mathcal{F})). \tag{3.12}$$



## 4 Eigenvalues of the transversal Dirac operator

In this section, we prove Kirchberg-type inequalities by using the transversal Kählerian twistor operators on Kähler spin foliations. We refer to [Kir90], [Kir92].

**Definition 4.1** *On a Kähler spin foliation, we define the transversal Kählerian twistor operators by*

$$\mathcal{P}^{(r)} : \Gamma(S_r(\mathcal{F})) \xrightarrow{\nabla^{tr}} \Gamma(Q^* \otimes S_r(\mathcal{F})) \xrightarrow{\pi_r} \Gamma(\ker \mathcal{M}_r),$$

where  $\mathcal{M}_r$  is the transversal Clifford multiplication defined by

$$\begin{aligned} \mathcal{M}_r : \Gamma(Q^* \otimes S_r(\mathcal{F})) &\longrightarrow \Gamma(S_{r-1}(\mathcal{F})) \oplus \Gamma(S_{r+1}(\mathcal{F})) \\ X \otimes \Psi_r &\longmapsto p_-(X) \cdot \Psi_r \oplus p_+(X) \cdot \Psi_r. \end{aligned}$$

For all  $r \in \{0, \dots, m\}$  and  $\Psi_r \in \Gamma(S_r(\mathcal{F}))$ , we have

$$\mathcal{P}^{(r)} \Psi_r = \sum_{i=1}^q e_i^* \otimes (\nabla_{e_i} \Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+ \Psi_r + b_r p_+(e_i) \cdot \mathcal{D}_- \Psi_r), \quad (4.1)$$

where  $\mathcal{D}_\pm = D_\pm + \frac{1}{2}p_\pm(\kappa)$  with  $a_r = \frac{1}{2(r+1)}$  and  $b_r = \frac{1}{2(m-r+1)}$ . For any spinor field  $\Psi_r \in \Gamma(S_r(\mathcal{F}))$ , we can easily prove

$$\sum_{i=1}^q e_i \cdot \mathcal{P}_{e_i}^{(r)} \Psi_r = 0. \quad (4.2)$$

**Remark 4.2** *For any non zero eigenvalue  $\lambda$  of  $D_{tr}$ , there exists a spinor field  $\Psi \in \Gamma(S(\mathcal{F}))$  called of type  $(r, r+1)$ , such that  $D_{tr} \Psi = \lambda \Psi$  and  $\Psi = \Psi_r + \Psi_{r+1}$ , with  $r \in \{0, \dots, m-1\}$ . By using (3.10), (3.11) and (3.12) it follows that  $D_- \Psi_r = D_+ \Psi_{r+1} = 0$ ,  $D_- \Psi_{r+1} = \lambda \Psi_r$ ,  $D_+ \Psi_r = \lambda \Psi_{r+1}$  and  $\|\Psi_r\|_{L^2} = \|\Psi_{r+1}\|_{L^2}$ .*

**Proof.** Let  $\varphi$  be an eigenspinor of  $D_{tr}$ . There exists an  $r$  such that  $\varphi_r$  does not vanish. Let  $\Psi = \frac{1}{\lambda} D_- D_+ \varphi_r + D_+ \varphi_r$ , one can easily get that  $D_{tr} \Psi = \lambda \Psi$ .

**Theorem 4.3** *Let  $(M, g_M, \mathcal{F})$  be a compact Riemannian manifold with a Kähler spin foliation  $\mathcal{F}$  of codimension  $q = 2m$  and a bundle-like metric  $g_M$*

with  $\kappa \in \Omega_B^1(\mathcal{F})$  and  $\delta_B \kappa = 0$ . Then any eigenvalue  $\lambda$  of the transversal Dirac operator, satisfies

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla. \quad (4.3)$$

If  $\Psi$  is an eigenspinor of type  $(r, r+1)$  associated with an eigenvalue  $\lambda$  satisfying equality in (4.3), then  $r = \frac{m-1}{2}$ , the foliation  $\mathcal{F}$  is minimal and for all  $X \in \Gamma(Q)$ , the spinor  $\Psi$  satisfies

$$\nabla_X \Psi + \frac{\lambda}{2(m+1)} (X \cdot \Psi - i\varepsilon J(X) \cdot \bar{\Psi}) = 0, \quad (4.4)$$

where  $\varepsilon = (-1)^{\frac{m-1}{2}}$ , and  $\bar{\Psi} := (-1)^r (\Psi_r - \Psi_{r+1})$ . As a consequence  $m$  is odd and  $\mathcal{F}$  is transversally Einstein with non negative constant transversal curvature  $\sigma^\nabla$ .

**Proof.** For all  $\Psi_r \in \Gamma(S_r(\mathcal{F}))$ , using Identities (4.1) and (4.2), we have

$$\begin{aligned} |\mathcal{P}^{(r)} \Psi_r|^2 &= \sum_{i=1}^q |\mathcal{P}_{e_i}^{(r)} \Psi_r|^2 = \sum_{i=1}^q (\mathcal{P}_{e_i}^{(r)} \Psi_r, \nabla_{e_i} \Psi_r) \\ &= \sum_{i=1}^q (\nabla_{e_i} \Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+ \Psi_r \\ &\quad + b_r p_+(e_i) \cdot \mathcal{D}_- \Psi_r, \nabla_{e_i} \Psi_r). \end{aligned}$$

Finally we obtain,

$$|\mathcal{P}^{(r)} \Psi_r|^2 = |\nabla^{tr} \Psi_r|^2 - a_r |\mathcal{D}_+ \Psi_r|^2 - b_r |\mathcal{D}_- \Psi_r|^2. \quad (4.5)$$

Let  $\lambda$  be an eigenvalue of  $D_{tr}$  and let  $\Psi$  an eigenspinor of type  $(r, r+1)$ . Applying Equality (4.5) to  $\Psi_r$ , one gets

$$\begin{aligned} |\mathcal{P}^{(r)} \Psi_r|^2 &= |\nabla^{tr} \Psi_r|^2 - a_r \lambda^2 |\Psi_{r+1}|^2 - a_r \lambda \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) \\ &\quad - \frac{a_r}{4} |p_+(\kappa) \cdot \Psi_r|^2 - \frac{b_r}{4} |p_-(\kappa) \cdot \Psi_r|^2. \end{aligned}$$

By the Schrödinger-Lichnerowicz formula and by the fact that  $\Psi_r$  and  $\Psi_{r+1}$  have the same  $L^2$ -norms, we get

$$\begin{aligned} \int_M |\mathcal{P}^{(r)} \Psi_r|^2 + \frac{a_r}{4} \int_M |p_+(\kappa) \cdot \Psi_r|^2 + \frac{b_r}{4} \int_M |p_-(\kappa) \cdot \Psi_r|^2 &= \\ \int_M ((1 - a_r) \lambda^2 - \frac{1}{4} K_\sigma^\nabla) |\Psi_r|^2 - a_r \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r). \end{aligned} \quad (4.6)$$

Similarly applying (4.5) to  $\Psi_{r+1}$ , we obtain

$$\begin{aligned} \int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4} \int_M |p_+(\kappa) \cdot \Psi_{r+1}|^2 + \frac{b_{r+1}}{4} \int_M |p_-(\kappa) \cdot \Psi_{r+1}|^2 = \\ \int_M ((1 - b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla)|\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r), \end{aligned} \quad (4.7)$$

where  $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$ . In order to get rid the term  $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r)$ , since the l.h.s. of (4.6) and (4.7) are non negative, dividing (4.6) by  $a_r$  and (4.7) by  $b_{r+1}$  then summing up, we find by substituting the values of  $a_r$  and  $b_{r+1}$ ,

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla.$$

Now, we discuss the limiting case of Inequality (4.3). Dividing (4.6) by  $a_r$  and (4.7) by  $b_{r+1}$  then summing up as before, and substituting  $a_r$ ,  $b_{r+1}$  and  $\lambda^2$  by their values, we easily deduce that  $\kappa = 0$ ,  $\mathcal{P}^{(r)}\Psi_r = 0$  and  $\mathcal{P}^{(r+1)}\Psi_{r+1} = 0$ . Hence by (4.6), we find that  $\lambda^2 = \frac{1}{4(1-a_r)}\sigma_0 = \frac{m+1}{4m}\sigma_0$  where  $\sigma_0 = \inf_M \sigma^\nabla$ , then  $r = \frac{m-1}{2}$  and  $m$  is odd. It remains to prove that  $\Psi$  satisfies (4.4). For  $r = \frac{m-1}{2}$ , by definition of the Kählerian twistor operators, for all  $j \in \{1, \dots, q\}$ , we obtain

$$\nabla_{e_j}\Psi_r + \frac{\lambda}{m+1}p_-(e_j) \cdot \Psi_{r+1} = 0,$$

and

$$\nabla_{e_j}\Psi_{r+1} + \frac{\lambda}{m+1}p_+(e_j) \cdot \Psi_r = 0.$$

Summing up the two equations, we get (4.4) for  $X = e_j$ . Using Ricci identity in (4.4), one easily proves that  $\mathcal{F}$  is transversally Einstein.  $\square$

**Theorem 4.4** *Under the same conditions as in Theorem 4.3 for  $m$  even, any eigenvalue  $\lambda$  of the transversal Dirac operator satisfies*

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^\nabla. \quad (4.8)$$

*If  $\Psi$  is an eigenspinor of type  $(r, r+1)$  associated with an eigenvalue satisfying equality in (4.8), then  $r = \frac{m}{2}$ , the foliation  $\mathcal{F}$  is minimal and  $\Psi$  satisfies for all  $X \in \Gamma(Q)$ ,*

$$\nabla_X \Psi_{r+1} = -\frac{\lambda}{q}(X - iJX) \cdot \Psi_r. \quad (4.9)$$

**Proof.** Let  $\Psi$  an eigenspinor of type  $(r, r+1)$  associated with any eigenvalue  $\lambda$  of the transversal Dirac operator  $D_{tr}$ . Recalling Equalities (4.6) and (4.7), we have

$$0 \leq \int_M ((1 - a_r)\lambda^2 - \frac{1}{4}K_\sigma^\nabla)|\Psi_r|^2 - a_r\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r), \quad (4.10)$$

and

$$0 \leq \int_M ((1 - b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla)|\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r). \quad (4.11)$$

Hence if  $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) \leq 0$ , then by (4.11)

$$\lambda^2 \geq \frac{1}{4(1 - b_{r+1})}K_0^\nabla,$$

The antilinear isomorphism  $j$  sends  $S_r(\mathcal{F})$  to  $S_{m-r}(\mathcal{F})$ . This allows the choice of  $\mu_r$  to be non negative (i.e.  $r \geq \frac{m}{2}$ ) where  $\mu_r$  is the eigenvalue associated with  $\Psi_r$ . Then a careful study of the graph of the function  $\frac{1}{1-b_{r+1}}$ , yields (4.8).

On the other hand if  $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) > 0$ . Applying Equation (4.5) to the spinor  $j\Psi$ , which is a spinor of type  $(m - (r + 1), m - r)$ , we find the same inequalities as (4.10) and (4.11), then

$$\lambda^2 > \frac{1}{1 - a_r} \frac{K_0^\nabla}{4}.$$

As before we can choose  $\mu_{m-(r+1)} \geq 0$  (i.e.  $r \leq \frac{m}{2} - 1$ ). A careful study of the graph of the function  $\frac{1}{1-a_r}$  gives Inequality (4.8).

Now we discuss the limiting case of (4.8). As we have seen, it could not be achieved if  $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) > 0$ , so only the other case should be considered. By (4.7), one has

$$\begin{aligned} & \int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4} \int_M |p_+(\kappa) \cdot \Psi_{r+1}|^2 \\ & + \frac{b_{r+1}}{4} \int_M |p_-(\kappa) \cdot \Psi_{r+1}|^2 - b_{r+1}\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) = \\ & (1 - b_{r+1}) \int_M (\frac{m}{4(m-1)}K_0^\nabla - \frac{1}{4(1-b_{r+1})}K_\sigma^\nabla)|\Psi_{r+1}|^2. \end{aligned}$$

Since  $\frac{m}{m-1} = \inf_{r \geq \frac{m}{2}} \frac{1}{1-b_{r+1}}$ , and the l.h.s. of (4) is non negative, we deduce that  $\kappa = 0$ ,  $\mathcal{P}^{r+1}\Psi_{r+1} = 0$  and  $\frac{m}{m-1} = \frac{1}{1-b_{r+1}}$  so  $r = \frac{m}{2}$ . It remains to show that Equation (4.9) holds. For this, take  $X = e_j$  where  $\{e_j\}_{j=1, \dots, q}$  is a local orthonormal frame. For  $r = \frac{m}{2}$ , and by definition of the Kählerian twistor operators, for all  $j \in \{1, \dots, q\}$ , we obtain

$$\nabla_{e_j} \Psi_{r+1} + \frac{\lambda}{q}(e_j - iJe_j) \cdot \Psi_r = 0.$$

□

## References

- [AG97] A. El Kacimi Alaoui and B. Gmira, *Stabilité du Caractère Kählérien Transverse*, Israel J. Math **101** (1997), 323–347.
- [Fri80] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalar-Krümmung*, Math. Nach. **97** (1980), 117–146.
- [GK91a] J.F. Glazebrook and F.W. Kamber, *On Spectral Flow of Transversal Dirac Operators and a theorem of Vafa-Witten*, Ann. Glob. Anal. Geom. **9** (1991), 27–35.
- [GK91b] ———, *Transversal Dirac families in Riemannian foliations*, Commun. Math. Phys. **140** (1991), 217–240.
- [Hij94a] O. Hijazi, *Eigenvalues of the Dirac Operator On Compact Kähler Manifolds*, Commun. Math. Phys. **160** (1994), 563–579.
- [Hij94b] ———, *Twistor Operators and Eigenvalues of the Dirac Operator*, Proceedings of the conference on Quaternionic-Kähler Geometry (Trieste), 1994.
- [JK03] S.D. Jung and Tae Ho Kang, *Lower bounds for the eigenvalue of the transversal Dirac operator on a Kähler foliation*, J. Geom. Phys. **45** (2003), 75–90.
- [Jun01] S.D. Jung, *The first eigenvalue of the transversal Dirac operator*, J. Geom. Phys. **39** (2001), 253–264.
- [Kir86] K.D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Glob. Anal. Geom. **3** (1986), 291–325.
- [Kir90] ———, *The first eigenvalue of the Dirac operator on Kähler manifolds*, J. Geom. Phys. **4** (1990), 449–468.
- [Kir92] ———, *Properties of Kählerian twistor-spinors and vanishing theorems*, Math. Ann. **293** (1992), 349–369.
- [Kir96] ———, *Killing Spinors on Kähler Manifolds*, Ann. Glob. Anal. Geom. **11** (1996), 141–164.
- [Ton88] Ph. Tondeur, *Foliations on Riemannian manifolds*, Springer, New York, 1988.

- [YT90] S. Yorozu and T. Tanemura, *Green's theorem on a foliated Riemannian manifold and its applications*, Acta. Math. Hung. **56** (1990), 239–245.